Introduction

A signal is a physical quantity that varies with one or more variables and is utilised in a wide range of applications across various systems. These signals can be represented in the time domain or analysed in the frequency domain.

Signal processing is a technique that involves the manipulation, detection, and interpretation of signals carrying information. It is based on signal theory, which characterises these signals in terms of their various properties. The applications of signal processing are vast and include fields such as telecommunications, geophysics, pattern recognition, biomedical engineering, and acoustics, among others.

Signal processing can be performed using analogue components (electronic components for signals, optical components for images). However, after the sampling stage, digital processing techniques, including digital filters, offer significant advantages in terms of implementation simplicity and flexibility, allowing for adjustments that make them preferable to analogue filtering methods.

This manuscript introduces key concepts of biomedical signal processing used in the engineering field. It aims to present the technical principles of analogue and digital signal processing. The main topics covered include convolution, correlation, sampling, and the design and application of analogue and digital filters, particularly finite impulse response (FIR) and infinite impulse response (IIR) filters. The content is based on the bibliography provided at the end of the document;

readers are strongly encouraged to consult the cited works for further study.

1 Temporal and Frequency Analysis of Signals

1.1 Temporal Analysis of Signals

1.1.1 Introduction

A signal is a physical quantity that varies with one or more variables and is used in various applications across different systems. These systems may be affected by disturbing phenomena known as noise. Signals can be represented in the time domain or analyzed in the frequency domain.

1.1.2 Signal Classification

Signals can be classified based on the following criteria:

1 **Temporal Characteristics**: Signals can be categorized as either:

- **Continuous-Time Signal**: A signal defined for every instant of time.
- **Discrete-Time Signal**: A signal defined only at discrete time intervals.

2 **Deterministic or Random Signals**:

- **Deterministic Signal**: A signal that can be described by a specific mathematical function and can be predicted with certainty.
- **Random Signal**: A signal that cannot be expressed mathematically in a deterministic form and is characterized only by its statistical properties.

3 Signal Parity:

- **Even Signal**: A signal is considered even if it is symmetrical about the y-axis. It is defined by the relation:

$$x_p(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

- **Odd Signal**: A signal is considered odd if it is symmetrical about the origin. It is defined by the relation:

$$x_{imp}(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

4 Signal Causality:

- A signal is said to be causal if its values are zero for all negative times. In other words, a causal signal does not exist prior to *t=0.*

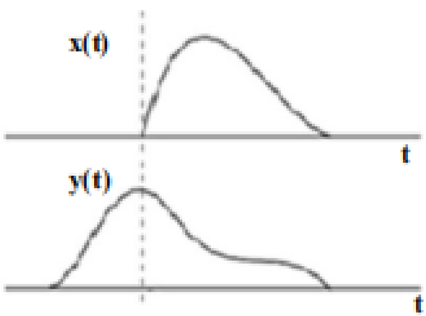

Figure 1. Causal Signal

5 Signals with Finite Energy / Power:

- **Energy**: A signal has finite energy if the total energy over all time is finite. The energy *Ex* of a signal *x(t)* is given by:

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

- **Power**: A signal has finite power if the average power over all time is finite. The average power *Px* is given by:

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

This classification can be summarized in the following figure:

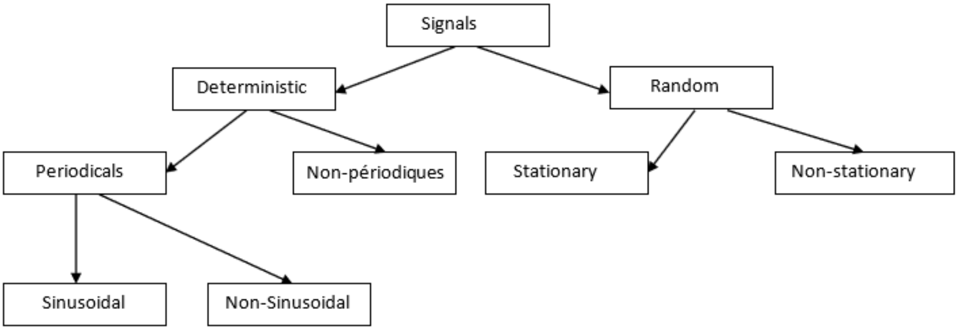

Figure 2. Signal Classification

1.1.3 Basic and Common Signals:

Here are some common basic signals:

1.1.3.1 Echelon (Unit Step) Function:

$$e(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases}$$

The unit step function is primarily used to model causality in signals, as multiplying a signal by a unit step function eliminates its negative part, making it causal.

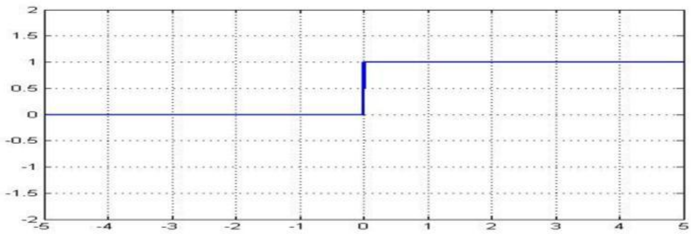

Figure 3. Echelon (Unit Step) Signal

1.1.3.2 Sign Function

$$e(t) = \begin{cases} 1, & \text{if } t > 0 \\ -1, & \text{if } t < 0 \end{cases}$$

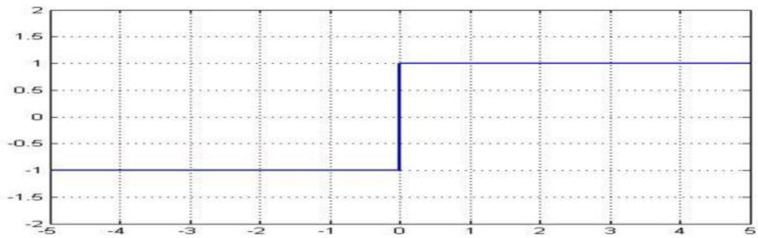

Figure 4. Sign Function

1.1.3.3 Triangular Function

$$\mathrm{tri}(t) = \begin{cases} 1 - |t|, & \text{if } |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

1.1.3.4 Ramp Function

$$\mathrm{ramp}(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

The ramp function is typically used to represent a gradually increasing signal.

1.1.3.5 *Rectangular Function*

$$\mathrm{rect}(t) = \begin{cases} 1, & \text{if } -2 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

A rectangular function can be used to isolate a segment of a signal over a specific duration.

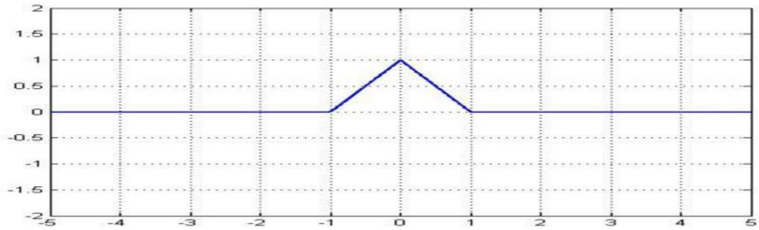

Figure 5. Rectangular Signal

A rectangular function of amplitude *A* and duration *T*, centered at t_0, is expressed as:

$$x(t) = A \cdot \mathrm{rect}(t - t_0)$$

The primary use of this function is to "window" a signal, isolating a specific segment of duration *T*. The operation is expressed as:

$$x_T(t) = x(t) \cdot \mathrm{rect}(t - t_0)$$

1.1.3.6 *Gaussian Pulse*

A Gaussian pulse is given by:

$$\mathrm{imp}_{\mathrm{Gauss}}(t) = e^{-\pi t^2}$$

This signal is often used in filtering and communication systems.

1.1.3.7 Dirac Delta Function (Impulse)

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases}$$

The Dirac delta function is used to model instantaneous impulses. It is the limit of a rectangular function as its width approaches zero while its amplitude approaches infinity.

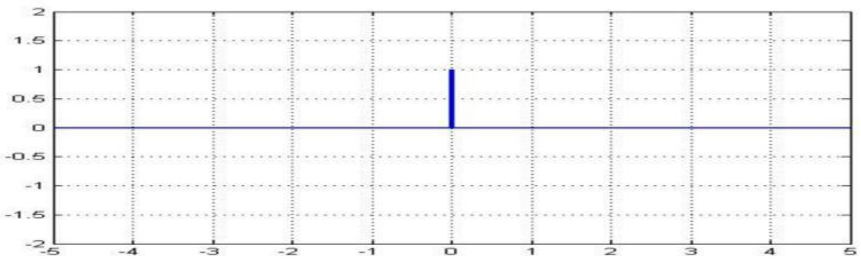

Figure 9. Dirac Impulse

The integral of the Dirac delta function is defined as:

$$\int_{-\infty}^{+\infty} s(t) \cdot \delta(t)\, dt = s(0)$$

This is used to extract the instantaneous value of a signal at the point of the impulse.

1.1.3.8 Impulse Train (Dirac Comb)

An impulse train is a periodic sequence of Dirac delta functions:

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

The impulse train is often used in sampling operations, which involves obtaining a sequence of samples at a sampling rate

$$f_e = \frac{1}{T}.$$

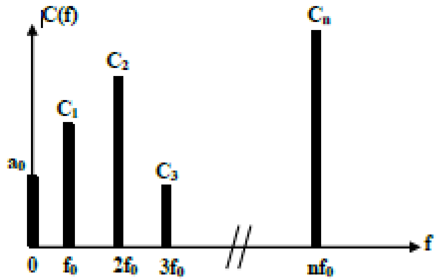

Figure 10. Impulse Train (Dirac Comb)

1.1.4 Convolution and Correlation of Signals

1.1.4.1 Convolution

1.1.4.1.1 Definition

Given two functions *f(t)* and *g(t)*, the convolution of these two functions is calculated by the following integral:

$$h(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t-\tau)\, d\tau$$

This integral reflects the overlap between the two functions *f(τ)* and *g(t−τ)*, which is inverted and shifted from the origin.

1.1.4.1.2 Convolution Properties

1 Commutativity

$$(x(t) * y(t)) = (y(t) * x(t))$$

2 Distributivity

$$[x(t) + y(t)] * z(t) = [x(t) * z(t)] + [y(t) * z(t)]$$

3 Associativity

$$[x(t) * y(t)] * z(t) = x(t) * [y(t) * z(t)]$$

4 Neutral Element

The neutral element of convolution is the Dirac delta function. It satisfies the following property:

$$\int_{-\infty}^{+\infty} \delta(\tau) \cdot x(t-\tau)\, d\tau = x(t)$$

Similarly, for a shifted delta function, we have:

$$\delta(t-t_0) * x(t) = x(t-t_0)$$

5 Differentiation

$$[x(t) * y(t)]' = x'(t) * y(t) = x(t) * y'(t)$$

6 **Plancherel's Theorem**:

Let *x(t)* and *y(t)* be two signals with Fourier transforms *X(f)* and *Y(f)*, respectively. Then the following relationships hold:

$$\mathcal{F}\{x(t) * y(t)\} = X(f) \cdot Y(f)$$

$$\mathcal{F}\{x(t) \cdot y(t)\} = X(f) * Y(f)$$

1.1.4.2 Correlation

1.1.4.2.1 Definition

Given two finite-energy signals *f(t)* and *g(t)*, the cross-correlation function $C_{fg}(\tau)$ is defined by the following integral:

$$C_{fg}(\tau) = \int_{-\infty}^{+\infty} f(t) \cdot g^*(t-\tau)\, dt$$

this dot product measures the degree of similarity in shape and position of the signals as they are shifted over time τ. $C_{fg}(\tau)$ represents the interaction energy between *f(t)* and $g(t-\tau)$. If $C_{fg}(\tau)=0$, then the two signals are decorrelated.

The autocorrelation function $C_{ff}(\tau)$ is a special case where the same signal is correlated with itself.

1.1.4.2.2 Correlation Properties

1 At $\tau=0$, the autocorrelation function is equal to the energy of the signal:

$$C_{xx}(0) = \int_{-\infty}^{+\infty} x(t) \cdot x(t)\, dt = \int_{-\infty}^{+\infty} x(t)^2\, dt = E_x$$

2 The cross-correlation function satisfies the following relationship:

$$C_{xy}(\tau) = C_{yx}^*(-\tau)$$

3 For a real-valued signal, the autocorrelation function is an even function

$$C_{xx}(\tau) = C_{xx}(-\tau)$$

4 For infinite-energy signals, the cross-correlation and autocorrelation functions are defined as:

$$C_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{+T} x(t) \cdot x(t-\tau)\, dt$$

Similarly, the cross-correlation between two infinite-energy signals *x(t)* and *y(t)* is:

$$C_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{+T} x(t) \cdot y(t-\tau)\, dt$$

5 Relationship between Correlation and Convolution

$$C_{xy}(\tau) = x(-\tau) * y(\tau)$$

1.1.4.3 Spectral and Interspectral Density

1.1.4.3.1 Spectral Energy Density (ESD)

Let *x(t)* be a real, finite-energy signal, and *X(f)* its Fourier transform. The spectral density $S_{xx}(f)$ of *x(t)* is defined as the square of the modulus of the Fourier transform:

$$S_{xx}(f) = |X(f)|^2$$

The total energy E_x of the signal can also be expressed as:

$$E_x = \int_{-\infty}^{+\infty} x(t)^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

The spectral energy density reflects the distribution of the energy of x(t)x(t)x(t) across different frequencies.